Two Walls and Two Ladders - A Calculus Problem

Jürgen Gerlach
jgerlach@radford.edu
Department of Mathematics and Statistics
Radford University
VA 24142
USA

Abstract
This paper deals with an extension of a classical Calculus problem: Two ladders are extended across two walls and joined at the top, and the sum of the lengths of the ladders is to be minimized. We derive a minimization problem for a function of two variables for this task and we inspect special cases. The software package Maple will be used in examples.

1 Introduction

This paper deals with the following problem:

A moat of width W is secured by two walls of height H_1 and H_2. Two ladders reach across the moat, one from each side, touching at the tips (see Figure 2). We seek to minimize the sum of the respective lengths of the ladders for this task.

This problem is discussed at great length by Edwards [1]. He derives a minimization problem in seven independent variables with five constraint equations, and the purpose of his tutorial is to illustrate the use of *Mathematica* for a larger Lagrange multiplier problem. In this article we reduce the problem to an optimization problem with two variables defined on a rectangular strip. Even with the aid of the computer algebra capabilities of Maple it is difficult to discuss the solution in its full generality. Instead we shall turn to selected special cases, and use the software for illustration.

2 A Wall, a Fence and a Single Ladder

In order to gain an understanding of the problem, we begin with a simpler case. This problem can be found in many Calculus texts: A fence of height H runs parallel to a wall at a distance W from the wall. What is the minimum length of a ladder reaching the wall across the fence?

This problem is equivalent to two other classical problems:

1. Given is a point (W, H) in the first quadrant. Of all lines passing through this point, find the one which minimizes the distance between the x-intercept and the y-intercept. Of course, the lines must have negative slopes, so that both intercepts exist and are positive.

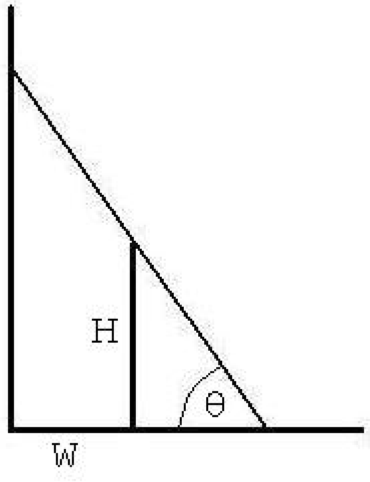

Figure 1: Single Ladder Problem

2. Two hallways of width W and H are joined at a right angle in an L shape. What is the length of the longest pipe which can be moved around this corner without moving the pipe vertically?

The solution to the single ladder problem is elementary. Let θ be the angle which the ladder makes with the horizontal, $0 < \theta < \frac{\pi}{2}$. Then the length of the ladder is

$$L = \frac{H}{\sin\theta} + \frac{W}{\cos\theta}$$

Setting $\dfrac{dL}{d\theta} = 0$ we arrive at

$$\tan^3\theta = \frac{H}{W} = \tan\alpha\,, \tag{1}$$

where α is the angle between the horizontal and the line connecting the origin to the top of the fence. If we substitute the optimal θ into the expression for L we see that

$$L_{min} = \left(H^{2/3} + W^{2/3}\right)^{3/2} \qquad \text{or, equivalently,} \qquad L_{min}^{2/3} = H^{2/3} + W^{2/3}$$

3 Derivation of the Equations

The physical setup and the parameters of the problem are described in Figure 2. We shall assume that $W > 0$ and that $0 < H_1 \leq H_2$, i.e. we always have the taller wall on the right, if the walls are uneven.

Let y denote the height above the ground of the point where the two ladders touch. Then we can express the total length of the ladders as

$$L = y\left(\frac{1}{\sin\theta} + \frac{1}{\sin\tau}\right)$$

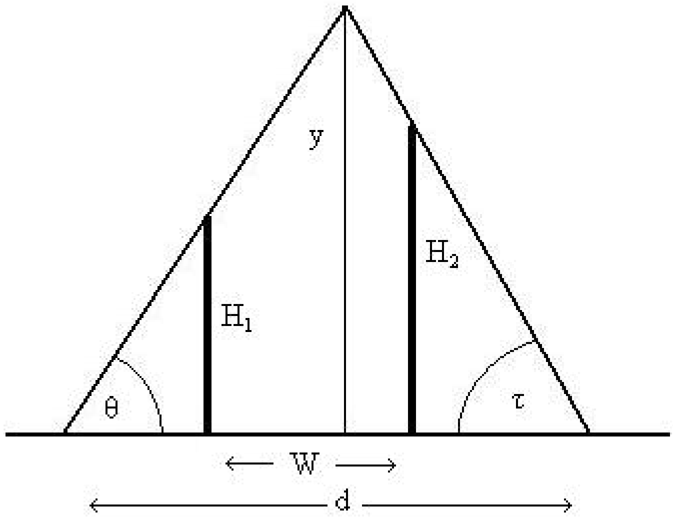

Figure 2: Two Ladder Problem

where θ and τ are the angles indicated in the figure. In the next step we shall calculate the distance d between the bases of the ladders in two ways. First, we consider the two right triangles which each have a ladder as their hypothenuse. We obtain

$$d \;=\; y\,(\cot\theta + \cot\tau)\;.$$

On the other hand, we can calculate d using the width W of the moat, along with the two smaller right triangles involving the heights of the respective walls. This approach yields

$$d \;=\; H_1\cot\theta + W + H_2\cot\tau\;.$$

Now we equate the two expressions for d and solve for y to obtain

$$y \;=\; \frac{W + H_1\cot\theta + H_2\cot\tau}{\cot\theta + \cot\tau}\;.$$

A final substitution yields an explicit formula for L in terms of the two angles.

$$L \;=\; \frac{W + H_1\cot\theta + H_2\cot\tau}{\cot\theta + \cot\tau}\left(\frac{1}{\sin\theta} + \frac{1}{\sin\tau}\right)$$

In order to simplify our notation further, we introduce the new variables

$$u = \cot\theta \qquad \text{and} \qquad v = \cot\tau \tag{2}$$

Then

$$y \;=\; \frac{W + H_1u + H_2v}{u+v} \tag{3}$$

$$L \;=\; y\,(\sqrt{1+u^2} + \sqrt{1+v^2})$$

$$\;=\; \frac{(W + H_1u + H_2v)(\sqrt{1+u^2} + \sqrt{1+v^2})}{u+v} \tag{4}$$

The problem at hand is to minimize this function L of two variables, which appears to be considerably simpler than minimizing a function of seven variables with five constraint equations. Before we address questions regarding the admissible domain of our function, we test our model using the data in Edwards' paper. The following can be reviewed in the Maple worksheet *Example1.mws*.

Example 1 *Let $H_1 = 10$, $H_2 = 15$ and let $W = 50$. Using Maple and its Optimization package we obtain the following result:*

```
restart; with(Optimization):
L := (u,v) -> (W+H[1]*u+H[2]*v)*(sqrt(1+u^2)+sqrt(1+v^2))/(u+v):
> L(u,v);

                                   2 1/2          2 1/2
     (W + H[1] u + H[2] v) ((1 + u )     + (1 + v )    )
     -------------------------------------------------
                           u + v

> W := 50; H[1] := 10; H[2] := 15; #Edwards data
                          W := 50
                         H[1] := 10
                         H[2] := 15
> Minimize(L(u,v));
  [102.715064501163539,
      [v = 0.874537802410209286, u = 1.72360788693863709]]
```

Recall that $u = \cot\theta$ and $v = \cot\tau$, and upon substitution, we find that $\theta \approx 30.1^o$ and $\tau \approx 48.8^o$. The total length of the two ladders adds up to $L \approx 102.7$. These results agree fully with those of Edwards.

Now we take a closer look at the feasible domain of the function L. It is obvious that the angles θ and τ must be positive in order to clear the respective walls, and their maximum is $\pi/2$. This, according to (2), translates into $u, v \geq 0$. Inspection of the formula (4) for L requires that we exclude the origin from the domain of L as well. $u = v = 0$ occurs when both ladders stand vertically.

There is a further restriction: Since y represents the height above ground of the point where the two ladders meet, its value must be at least the height of the taller of the two walls. In other words, we need to require that $0 < H_1 \leq H_2 \leq y$. In terms of the variables u and v this condition becomes

$$\begin{aligned} H_2 &\leq y = \frac{W + H_1 u + H_2 v}{u+v} \\ H_2(u+v) &\leq W + H_1 u + H_2 v \\ (H_2 - H_1)u &\leq W \\ u &\leq u_0 = \frac{W}{H_2 - H_1} \end{aligned} \tag{5}$$

with the understanding that there is no upper bound for u if $H_1 = H_2$. Geometrically the condition (5) can be interpreted as follows: The line connecting the tops of the two walls has slope $m = \frac{H_2 - H_1}{W}$, and the requirement that $\cot\theta = u \le u_0 = 1/m$ is equivalent to saying that the slope of the ladder on the left is at least m.

The function L defined by the equation (4) is smooth, except for the singularities along the line $u + v = 0$. The geometry of the problems reduces the feasible domain to the strip $0 \le u \le u_0$ and $v \ge 0$ without the origin. A minimum occurs at a stationary point or at the boundary of the region, and we need to compute the gradient of L in order to locate stationary points.

Beginning with the function y given by (3) we find that

$$\frac{\partial y}{\partial u} = \frac{(u+v)H_1 - (W + H_1 u + H_2 v)}{(u+v)^2} = -\frac{y - H_1}{u+v}$$

and similarly

$$\frac{\partial y}{\partial v} = -\frac{y - H_2}{u+v}$$

Since $L = y\left(\sqrt{1+u^2} + \sqrt{1+v^2}\right)$ we have

$$\begin{aligned}\frac{\partial L}{\partial u} &= y\,\frac{u}{\sqrt{1+u^2}} + \left(\sqrt{1+u^2} + \sqrt{1+v^2}\right)\frac{\partial y}{\partial u} \\ &= \frac{uy}{\sqrt{1+u^2}} - (y - H_1)\,\frac{\sqrt{1+u^2}+\sqrt{1+v^2}}{u+v}\end{aligned}$$

and

$$\frac{\partial L}{\partial v} = \frac{vy}{\sqrt{1+v^2}} - (y - H_2)\,\frac{\sqrt{1+u^2}+\sqrt{1+v^2}}{u+v}$$

Finally we introduce the auxiliary function $A = \dfrac{\sqrt{1+u^2}+\sqrt{1+v^2}}{u+v}$. Then

$$\frac{\partial L}{\partial u} = \frac{uy}{\sqrt{1+u^2}} - (y - H_1)\,A \tag{6}$$

$$\frac{\partial L}{\partial v} = \frac{vy}{\sqrt{1+v^2}} - (y - H_2)\,A \tag{7}$$

Even with the aid of computer algebra systems it is difficult to find a meaningful expression for the stationary points of L involving the parameters H_1, H_2 and W. As an alternative we shall investigate a few special cases.

4 Special Cases

4.1 The Symmetric Case - Walls Of Identical Height

Let us begin with the symmetrical case and set $H = H_1 = H_2$. This is equivalent to the single ladder case from Section 2 with W replaced by $W/2$. According to formula (1) we obtain

$$\tan^3\theta = \tan^3\tau = \frac{H}{W/2},$$

which translates into

$$u^3 \;=\; v^3 \;=\; \frac{W}{2H} \tag{8}$$

We leave it as an exercise to verify that the function L, which was derived in the previous section, has exactly one stationary point located at $(u, v) = \left(\sqrt[3]{W/(2H)}, \sqrt[3]{W/(2H)}\right)$.

The Maple worksheet *SymmetricalCase.mws* contains a confirmation, that the point found in (8) is a stationary point of L, and that it yields a local minimum according to the second derivative test.

4.2 Which Side has the Steeper Ladder?

We have seen above that when the walls are of equal height, the optimal solution is symmetric, and both ladders stand at the same angle. Now we assume that $H_1 < H_2$, i.e. the wall on the right is taller than the one on the left, and we want to decide which of the two ladders stands at a steeper angle.

Again, we look for the stationary points of L and make repeated use of (6) and (7).

First consider the case that $y = H_2$. Then $\frac{\partial L}{\partial v} = 0$ implies that $v = 0$. This implies that the ladder on the right stands vertically, and the steeper of the two ladders is clearly on the right.

If $H_2 < y$, we set both partials of L equal to zero, and when we solve the resulting equations for A, arrive at

$$\frac{uy}{(y - H_1)\sqrt{1+u^2}} \;=\; A \;=\; \frac{vy}{(y - H_2)\sqrt{1+v^2}}$$

After elimination of y in the numerators, the equation can be written as

$$\frac{u}{\sqrt{1+u^2}} \;=\; \frac{y - H_1}{y - H_2}\,\frac{v}{\sqrt{1+v^2}}$$

By assumption we have $H_1 < H_2 < y$, which implies that $\frac{y-H_1}{y-H_2} > 1$. Thus

$$\frac{u}{\sqrt{1+u^2}} \;>\; \frac{v}{\sqrt{1+v^2}}$$

The function $f(t) = \frac{t}{\sqrt{1+t^2}}$ is strictly increasing, and hence it follows that $u > v$. But u and v are the cotangents of the respective angles as given by (2), and our result is equivalent to $\theta < \tau$. We conclude that the angle on the right must be greater than the angle on the left. In other words, the side with the taller wall will have the steeper ladder.

4.3 Solutions With Vertical Ladders?

Finally we investigate the possibility that an optimal solution contains a vertical ladder. We assume $H_1 < H_2$. Then the domain of L is the rectangular strip of points (u, v) with $0 \le u \le u_0 = \frac{W}{H_2 - H_1}$ and $v \ge 0$, excluding the origin. Asking for solutions with vertical ladders is equivalent to looking for optima on the boundaries $u = 0$ or $v = 0$.

In the previous section we have seen that stationary points occur in the region where $u > v$, which reduces the location of possible minima to the triangle in the uv-plane with vertices at $(0, 0)$, $(u_0, 0)$ and (u_0, u_0). Therefore the v-axis can be ruled out as location of an optimal solution, and we cannot have a vertical ladder on the left hand side.

Let us take a look at $\frac{\partial L}{\partial v}$ along the u-axis. Since $v = 0$ the term y becomes

$$y \;=\; \frac{W + H_1 u}{u} \;=\; H_1 \;+\; \frac{W}{u}\,.$$

Hence $y - H_2 \;=\; -(H_2 - H_1 - \frac{W}{u})$, and from (7) we obtain

$$\begin{aligned}\left.\frac{\partial L}{\partial v}\right|_{v=0} \;&=\; -(y - H_2)\,\frac{1+\sqrt{1+u^2}}{u}\\ &=\; \left(H_2 - H_1 - \frac{W}{u}\right)\,\frac{1+\sqrt{1+u^2}}{u}\end{aligned}$$

We see that $\left.\frac{\partial L}{\partial v}\right|_{v=0}$ changes sign when $u = \frac{H_2-H_1}{W} = u_0$, and that $\frac{\partial L}{\partial v} < 0$ for $0 < u < u_0$. Therefore, L can be reduced by increasing v, i.e. by stepping into the interior of the domain, and we cannot have a minimum along this boundary line segment. Thus, an optimal solution involving a vertical ladder can only occur at the point $(u_0, 0)$.

We note that on the remaining boundary line $u = u_0$ of our domain we have $y = H_2$, as a brief computation shows. Moreover,

$$L(u_0, v) \;=\; H_2\,\left(\sqrt{1+u_0^2} + \sqrt{1+v^2}\right)$$

which is an increasing function of v. In this case the left ladder rests on the two walls, and the two ladders meet on the top of the right wall; hence $y = H_2$. If we manipulate the right ladder, the shortest ladder is obtained in the vertical position, i.e. when $v = 0$.

At this point we know that if a vertical ladder solution exists, it will occur at the point $(u_0, 0)$, but does this ever happen? The function L is differentiable, except along the line $u + v = 0$, and the domain restrictions were forced by the geometry of the problem and not by the smoothness of L. Thus we may take directional derivatives at the point $(u_0, 0)$ toward the interior of the feasible domain. Since $\left.\frac{\partial L}{\partial v}\right|_{(u_0,0)} = 0$ for any selection of the parameters H_1, H_2 and W, the answer to our investigation hinges on the sign of $\left.\frac{\partial L}{\partial u}\right|_{(u_0,0)}$, the u-component of the gradient of L. If this quantity is positive, we can further reduce L by moving into the interior of the admissible domain, otherwise we have an optimal solution at the boundary point $(u_0, 0)$.

According to (6) we have

$$\left.\frac{\partial L}{\partial u}\right|_{(u_0,0)} \;=\; \frac{H_2 u_0}{\sqrt{1+u_0^2}} \;-\; (H_2 - H_1)\,\frac{1+\sqrt{1+u_0^2}}{u_0}$$

Since $u_0 = \frac{W}{H_2-H_1}$, the sign of this partial derivative depends entirely on the data H_1, H_2 and W. If W is small in relation to $H_2 - H_1$, the term u_0 is small as well, and the partial derivative is negative, due to the huge contribution of the second term. In this case $(u_0, 0)$ is an optimal solution. Solving $\left.\frac{\partial L}{\partial u}\right|_{(u_0,0)} = 0$ for u_0 results in

$$u_0 \;=\; \sqrt{\left(\frac{H_2}{H_1}\right)^2 - 1} \tag{9}$$

Substitution of this result into (5) and solving for W leads to the definition of the critical width W_0 as

$$W_0 \;=\; (H_2 - H_1)\sqrt{\left(\frac{H_2}{H_1}\right)^2 - 1} \tag{10}$$

Our analysis shows that $(u_0, 0)$ is an optimal solution if $W \le W_0$. Otherwise L can be reduced by moving into the interior of the admissible domain.

In terms of the original problem we conclude that vertical ladders are coupled with narrow moats. In this case the left ladder rests on the two walls and the right ladder stands in a vertical position.

We conclude this section with an example.

Example 2 *We keep $H_1 = 10$ and $H_2 = 15$ as in Example 1. In this case the critical width from formula (10) becomes*

$$W_0 \;=\; 5\sqrt{5}/2 \;\approx\; 5.590$$

In Example 1 we found a solution for $W = 50$, and now we are interested in results for W near W_0. By gradually reducing W we shall observe how the stationary point migrates from the interior of the domain through the lower right corner to the outside of the domain.

We begin by taking $W = 8$. In this case $u_0 = 1.6$ and a numerical calculation in Maple locates the stationary point at approximately $(1.154, 0.191)$, which is well within the domain. The contour plot in Figure 3 shows a few level curves of L this case.

Next we set $W = W_0 = 5\sqrt{5}/2$. Here $u_0 = \sqrt{5}/2$, and the optimal solution is located at the corner of the domain, which is nicely illustrated by the graph of the level curves in Figure 4.

Finally we use $W = 5$, which implies that $u_0 = 1$. The stationary point is located at about $(1.130, -0.110)$, which is not part of the admissible domain, as shown in Figure 5. Thus the optimal solution is $(u, v) = (1, 0)$.

The Maple calculations for this example can be found in the file Example2.mws.

5 Conclusion

In this paper we studied a generalization of a classic Calculus problem. By selecting appropriate variables in the modeling process - cotangents of angles in this case - we arrived at a minimization problem with two independent variables. The problem is very intriguing. The physical setting is easily understood, yet a full general analysis in terms of the three parameters becomes difficult. We were able to characterize solutions for special cases and explain the occurrence of solutions with vertical ladders. Computer algebra software assisted with some of the calculations, it found approximate solutions in the examples, and its graphing capabilities provided a better understanding of the numerical results. The author would like to thank the anonymous referees for their helpful suggestions.

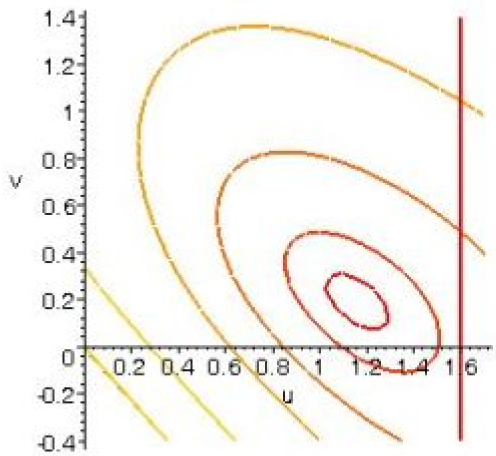

Figure 3: W=8

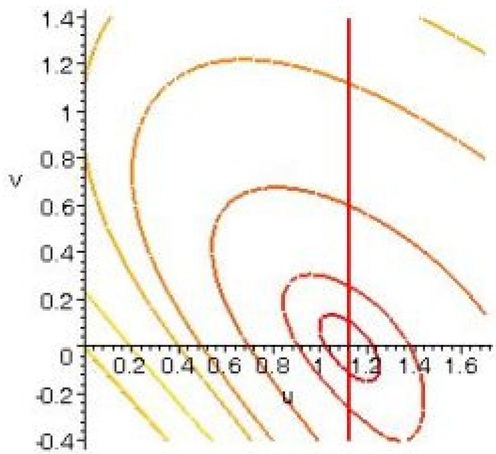

Figure 4: $W = 5\sqrt{5}/2$

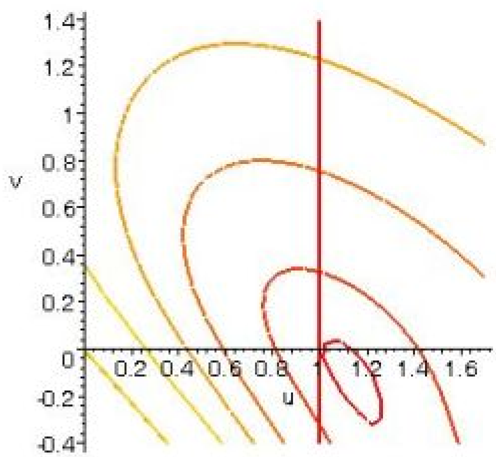

Figure 5: $W = 5$

References

[1] D. Henry Edwards, *Ladders, Moats, and Lagrange Multipliers*, The Mathematica Journal 4 (1994), p. p. 48-52.

Software Packages

- Maple 11.00, a product of Maplesoft, 2007.
  `http://www.maplesoft.com/`

Supplemental Electronic Materials

Maple files for this paper:

- *Example 1:* Calculation of the minimizer in Example 1 using the Optimization package.
  `Example1.mws`.
- *Example 2:* Calculation of the minimizers in Example 2 with the Optimization package for the widths $W = 8$, $W = W_0$ and $W = 5$ and generation of the associated level curves.
  `Example2.mws`.
- *The Symmetrical Case:* Confirmation that the solution found in Section 4.1 yields a minimum.
  `SymmetricalCase.mws`.